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OVERDETERMINED MATHEMATICAL PROBLEMS AT MATHEMATICS COMPETITIONS

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Abstract

The best way to develop mathematical culture among students and their active participation in mathematics education is the effective organization and management of the teaching process white solving mathematical problems. Through this, students consciously and permanently acquire mathematical knowledge, skills, and habits. This underscores the importance of problems in mathematics teaching, as well as the teacher's competence in effectively organizing and managing the teaching process during solving mathematical problems. In this paper, we are focused on one of the possible inconsistencies in formulating mathematical problems, which may be a real problem in effective use of mathematical problems for achieving the goals of mathematics education.

Keywords: *mathematics competition, mathematical problem, overdetermined problem.*

INTRODUCTION

A common requirement in mathematics is: *From a given set of mathematical objects* U *to separate a subset* S*, whose elements satisfy the given properties.* We say that this requirement defines a *mathematical problem* or simply a *problem*. More precisely, a mathematical problem is understood as a sequence of thoughts by which, in some way, a subset S of a given set U of mathematical objects is defined, requiring:

- if the set S is finite, find it constructively, or
- prove that S is a subset of an already defined subset of the set U, or
- prove that $S = S'$, where S' is a given set, or
- prove that the elements of S can be obtained by a finite application of the compass-and-ruller rules.

For each of the ways of defining S we say that it is a *representation* of S, and for the problem we say it is *determined* in the set U. The set S is said as the *solution set* of the problem in U.
The concept of a problem is related with the following terms:

- *the text of the problem* is the sequence of words from natural language and mathematical symbols which define the set U, the properties of elements in S and the requirement for representing S,
- *the condition of the problem* (A) is the text by which S is defined descriptively, and U is defined descriptively or constructively,
- *the conclusion of the problem* (B) is the part of the problem that indicates the required representation of S,
- *solving the problem* is the activity whereby from the given definition of S in the text one proceeds toward the required representation of S. The result of this activity is a finite

sequence of ways of defining S, which enables obtaining the required representation of S and each such representation is said to be a *solution of the problem* (R), and

- *the basis of the solution of the problem* (C) is the set of theorems, axioms, definitions, etc., and the solution is obtained by their application.

There are several classifications of mathematical problems, but the subject of our discussion will be the classification of mathematical problems according to the condition of the problem. According to *the condition,* problems are classified as *determined*, *underdetermined, and overdetermined*.

- A problem and informations which the condition (A) contains sufficient and only sufficient data to satisfy the conclusion (B) is called a *determined problem.*
- If from the condition (A) of some determined problem some information is omitted, we say we have an *underdetermined problem.*
- If, on the other hand, in the condition (A) of some determined problem a information is added that was not in (A), then an *overdetermined problem is obtained.* If the new information(condition) follows from the condition (A), then the new problem remains determined; but if the new datum contradicts the condition (A), then the new problem has no solution and we say we have a , the new problem has no solution and we say we have a *contradictory problem.*

SEVERAL OVERDETERMINED PROBLEMS FROM THE KANGAROO WITHOUT BORDERS COMPETITION

Overdetermined problems are very rare at mathematics competitions, but they are not entirely excluded. In the following, we will present three such problems given in the Kangaroo Without Borders international competition, all posed in different years in the Cadet category.

1. (Kangaroo Without Borders 2017). Four cousins Emma, Iva, Renata, and Zora are 3, 8, 12 and 14 years old, not necessarily to correspond. Emma is younger than Renata. The sum of the ages of Zora and Emma is divisible by 5. The sum of the ages of Zora and Renata is also divisible by 5. How old is Iva?

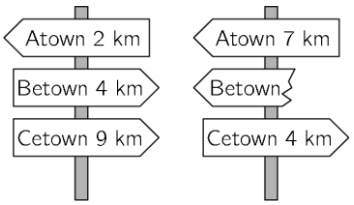
A) 14 B) 12 C) 8 D) 3

E) cannot be determined

Solution. A). Among the numbers 3, 8, 12 and 14 there are only two pairs of girls whose ages sum is a multiple of 5, namely: $3+12=15$ and $8+12=20$. Since one possibility belongs to Zora and Emma, and consecutively to Zora and Renata, we conclude that Emma, Zora, and Renata are 3, 8, and 12 years old. Therefore, Iva is 14 years old.

Comment. As we can see when solving the problem, the condition *Emma is younger than Renata* was not used, meaning it is unnecessary. Therefore, the problem is overdetermined. ■

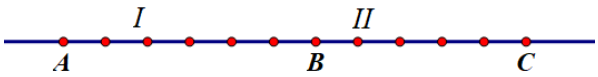
2. (Kangaroo Without Borders 2020). The shortest route from Atown to Cetown passes through Betown. Two signs are placed along the road from Atown to Cetown. What is the distance written on the broken sign?



A) 1 *km* B) 3 *km* C) 4 *km*

D) 5 *km* E) 9 *km*

Solution. A). We represent the positions of the cities and signs on the number line. The distance between any two points is 1 *km*.



From the road signs, we conclude that city town (A) is to the left of signs I and II. If we place city A at the leftmost point, then signs I and II are at 2 and 7 units of distance, respectively. Now city Betown (B) is four units to the right of sign I, meaning the distance between cities Atown and Betown is $2+4=6\ km$. But sign II is 7 *km* from city A town, so city Betown is between sign II and city A town and is at distance $7-6=1\ km$ from sign II. Therefore, the distance written on the broken sign is 1 *km*.
Comment. As we can see, while solving the problem we did not use any information about city Cetown. The problem can also be solved using only the information about cities Betown and Cetown, i.e., without using the information about city Atown. Therefore, the given problem is overdetermined. ■

3. (Kangaroo Without Borders 2015 year). A train consists of a locomotive and wagons. Each wagon has the same number of compartments. Pavel is traveling in the third wagon in the 18th compartment, counting after the locomotive. Matea is traveling in the seventh wagon in the 50th compartment, again counting again after the locomotive. How many compartments does each wagon have?
A) 7 B) 8 C) 9 D) 10 E) 12

Solution. B). Let x be the number of compartments in one wagon. Since Matea is in the seventh wagon in the 50th compartment, we get $50=6x+z$, where z is the ordinal number of the compartment in the seventh wagon where Matea is. Clearly, z is a natural number less than or equal to x. The obtained equation implies $6x<50=6x+z\le 7x$, therefare $\frac{50}{7}\le x<\frac{50}{6}$. But x is a natural number, so $x=8$. Therefore, each wagon has 8 compartments.
Comment. As we can see, while solving the problem we did not use the condition that Pavel is traveling in the third wagon in the 18th compartment counting after the locomotive. This means this condition can be omitted and the problem remains well-defined. ■

By solving the above problems, we established that they are overdetermined and also identified which condition is unnecessary, i.e., which condition should be omitted from the formulation.

OVERDETERMINED PROBLEMS CAN HAVE SIGNIFICANT DIDACTIC VALUE

The previously examined problems can be clearly and appropriately corrected. That would reduce the need for the students to engage in additional analysis that only complicates the problem-solving process and requires more time. It is evident that these examples do not allow for further exploration, and thus they lack significant didactic value.

However, there exist examples of overdetermined problems whose elaboration can be used to improve the quality of mathematical thinking and to enhance students' problem-solving abilities. In the following, we will examine one such problem published in the journal *Mathematical Paper* (XLIV-2), by the Society of Mathematicians of the Republic of Serbia. In the subsequent issue, the provided solution applies all the conditions stated in the problem. We will present it in full, with only minor corrections.

4. The lengths of the legs of a right triangle are positive integers (in centimeters). If the hypotenuse is $20\ cm$, the radius of the inscribed circle is $4\ cm$, and the legs differ by $4\ cm$, find the perimeter of this triangle.

Solution. Let a and b be the lengths of the legs, and c be the hypotenuse of the triangle. From the conditions of the problem it follows that $a = b + 4$.
Furthermore, for the area of the triangle we have

$$P = \frac{ab}{2} = \frac{b(b+4)}{2} \text{ and } P = \frac{a+b+c}{2} r = \frac{(b+4)+b+20}{2} \cdot 4 = 4b + 48,$$

from which we obtain $\frac{b(b+4)}{2} = 4b + 48$, that is $b^2 - 4b = 96$. From the last equation we obtain $b(b-4) = 96$. Since the side lengths are integers and $1 \cdot 96 = 2 \cdot 48 = 3 \cdot 32 = 4 \cdot 24 = 6 \cdot 16 = 8 \cdot 12 = 96$ the unique solution is $b = 12\ cm$. Therefore, $a = b + 4 = 16\ cm$ and $L = a + b + c = 48\ cm$. ■

In the formulation of the above problem the followings are given:

1) the triangle is right-angled,
2) the lengths of the legs are positive integers,
3) the length of the hypotenuse equals $20\ cm$,
4) the lengths of the legs differ by $4\ cm$ and
5) the radius of the inscribed circle is $4\ cm$,

meaning five informations (conditions) are given. Furthermore, in the proposed solution the equation $b^2 - 4b = 96$ is obtained, which is equivalent to $(b-2)^2 - 10^2 = 0$, i.e., to the equation $(b+8)(b-12) = 0$, from which $b = 12\ cm$ follows. Therefore, the condition i.e. 2) — the leg lengths are positive integers — is not necessary. This means that the condition makes the problem overdetermined, something that the problem's author, the journal's editors, the teachers, and the students did not notice. But is this the only information that makes the problem overdetermined? As we know, a triangle is determined when three of its independent elements are known, which means there is one more information, i.e., the problem is overdetermined on multiple grounds. In what follows, we give several reformulations of this problem, each omitting information 2) and one of the other four information.

5. Given a right triangle with hypotenuse $20\ cm$. If the legs differ by $4\ cm$, find the perimeter of this triangle.

Solution. Let a and b be the lengths of the legs, and c be the hypotenuse of the triangle. From the conditions it follows that $a = b + 4$. Now, by the Pythagorean theorem,

$$(b+4)^2 + b^2 = 20^2, \text{ i.e., } 2b^2 + 8b + 16 = 400.$$

The last equation is successively equivalent to

$$b^2 + 4b - 192 = 0,$$
$$(b+2)^2 - 14^2 = 0,$$
$$(b-12)(b+16) = 0.$$

A product of two numbers equals zero if and only if at least one of the factor equals to zero.Therefore, $b - 12 = 0$ or $b + 16 = 0$, that is $b = 12$ or $b = -16$. But siuce $b > 0$, the unique solution is $b = 12\ cm$. Therefore,

$$a = b + 4 = 16\ cm \text{ and } L = a + b + c = 48\ cm. \blacksquare$$

6. The lengths of two sides of triangle ABC differ by $4\ cm$, the length of the third side is $20\ cm$ and the radius of the inscribed circle of triangle ABC is $4\ cm$. Find the perimeter of this triangle.

Solution. Let a and b be the lengths of two sides of the triangle, and $c = 20\ cm$ be the length of its third side. From the given conditions it follows that $a = b + 4$. Furthermore, the semi-perimeter of the triangle is $s = \frac{a+b+c}{2} = \frac{(b+4)+b+20}{2} = b + 12$, so the area of the triangle is $P = sr = 4s = 4(b+12)$. On the other hand, by Heron's formula the area of the triangle is

$$\begin{aligned} P &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{(b+12)(b+12-b-4)(b+12-b)(b+12-20)} \\ &= \sqrt{96(b+12)(b-8)}. \end{aligned}$$

Now, using $b + 12 > 0$, from the last two equalities we successively obtain

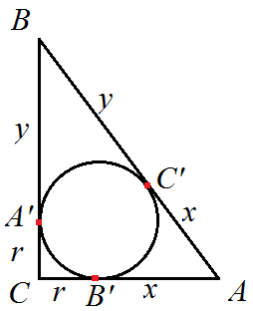
$$\begin{aligned} &4(b+12) = \sqrt{96(b+12)(b-8)}, \\ &16(b+12)^2 = 96(b+12)(b-8), \\ &b+12 = 6(b-8) \\ &5b = 60, \\ &b = 12\ cm. \end{aligned}$$

Therefore, $a = b + 4 = 16\ cm$ and $L = a + b + c = 48\ cm$. ■

7. Let ABC be a right triangle with hypotenuse $20\ cm$. If the radius of the inscribed circle is $4\ cm$, find the perimeter of this triangle.

Solution. Let C', B', A' be the points where the inscribed circle meets to the hypotenuse AB and the legs AC, BC of the right triangle ABC (see the figure). Corresponding then the properties of tangent segments imply

$\overline{AB'} = \overline{AC'} = x,\ \overline{BC'} = \overline{BA'} = y,\ \overline{CA'} = \overline{CB'} = r$.



Furthermore,

$$a = y + r, b = x + r, c = x + y,$$

therefore

$$\begin{aligned} L &= a + b + c = (y + r) + (x + r) + c \\ &= 2r + (x + y) + c = 2r + 2c \\ &= 2 \cdot 4 + 2 \cdot 20 = 48\ cm, \end{aligned}$$

So, the perimetar is 48cm. ■

8. Let ABC be a right triangle with inscribed circle radius $4\ cm$, in which the legs differ by $4\ cm$. Find the perimeter of this triangle.

Solution. Let C', B', A' be the points where the inscribed circle meets the hypotenuse AB and the legs AC, BC of the right triangle ABC (see the figure). Corresponding then from the properties of tangent segments imply $\overline{AB'} = \overline{AC'} = x,\ \overline{BC'} = \overline{BA'} = y,\ \overline{CA'} = \overline{CB'} = r$. Furthermore, $a = b + 4$, $c = x + y = b - r + a - r = b - 4 + b + 4 - 4 = 2b - 4$. Now, by the Pythagorean theorem we obtain the following

$$a^2+b^2=c^2,$$
$$(b+4)^2+b^2=(2b-4)^2,$$
$$2b^2+8b+16=4b^2-16b+16,$$
$$b^2=12b.$$

But $b>0$, therefore $b=12\ cm$. So,

$$a=12+4=16\ cm,\ c=2\cdot 12-4=20\ cm \text{ and } L=a+b+c=48\ cm. \blacksquare$$

Next, we consider one more problem obtained from Problem 4 by omitting two conditions (not including condition 2), whose solution is the same as the solutions of the previous problems.

9. The lengths of the legs of a right triangle are positive integers (in centimeters). If the hypotenuse is $20\ cm$, find the perimeter of this triangle.

Solution. Let a and b be the legs and $c=20\ cm$ be the hypotenuse of the right triangle ABC. By the Pythagorean theorem it follows that

$$a^2+b^2=20^2.$$

Now, since the lengths of the legs a and b are positive integers, without loss of generality we may assume that $a<b$. Accordingly, the ordered triple of natural numbers $(a,b,20)$ is a Pythagorean triple. Using the fact that the unique Pythagorean triple whose largest element is 20 is the triple $(12,16,20)$, we obtain $a=12$, $b=16$ and $L=a+b+c=48\ cm$. $\blacksquare$

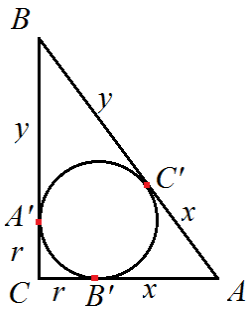
In the formulations of the previous five problems, we simply omitted two conditions from the formulation of Problem 4 and obtained problems that are not overdetermined and have the same solution as Problem 4. In the following, by omitting two conditions from Problem 4 we obtain a problem in which one of the solutions is the solution of Problem 4.

10. The lengths of the legs of a right triangle are positive integers (in centimeters). The radius of the inscribed circle of the triangle is $4\ cm$. Find the perimeter of this triangle.

Solution. Let a and b be the legs, c be the hypotenuse and $r=4\ cm$ be the radius of the inscribed circle of the right triangle ABC. Let $a=y+4$, $b=x+4$ and $c=x+y$. So, x,y are natural numbers greater than 4 and, without loss of generality, we may assume that $y>x$. Now, by the Pythagorean theorem follows the equation

$$(y+4)^2+(x+4)^2=(x+y)^2,$$

which is equivalent to $(x-4)(y-4)=32$. Furthermore, the given conditions and the last equation is equivalent to the following systems of equations



$$\begin{cases} x-4=1 \\ y-4=32 \end{cases} \qquad \begin{cases} x-4=2 \\ y-4=16 \end{cases} \qquad \begin{cases} x-4=4 \\ y-4=8 \end{cases}$$

whose solutions are, respectively, $x=5,\ y=36$; $x=6,\ y=20$ and $x=8,\ y=12$. Therefore, the problem has three possible solutions:

a) $a=40\ cm,\ b=9\ cm,\ c=41\ cm$ and $L=90\ cm$,

b) $a = 24\ cm,\ b = 10\ cm,\ c = 26\ cm$ and $L = 60\ cm$,

c) $a = 16\ cm,\ b = 12\ cm,\ c = 20\ cm$ and $L = 48\ cm$. ■

CONCLUSION

In this paper are presented four overdetermined problems, three of which were posed at mathematics competitions for students of the eighth and ninth grade in nineyear primary education and one in a mathematics journal. The three problems were published after the mentioned competitions were finished and the solution of the fifth problem is given in the next issue of the same journal. Furthermore, for none of the problems was a correction issued, which leads us to conclude that in all cases neither the teachers nor the students noticed that the problems are overdetermined.

It is natural and logical to ask what causes this situation, particularly in the case of Problem 4, for which it is obvious that it is overdetermined, since a triangle is determined by three independent elements. As far as we know, there is no existing research on this topic in the field of methodics of mathematics and mathematics education methodology. Nevertheless, we believe that errors of this kind stem from an incomplete realization of the steps of solving a mathematical problem, and in particular from a lack of thorough understanding of the problem and the additional analysis after it is solved, i.e., the first and fourth step. Furthermore, the presence of overdetermined problems can be an obstacle to students' acquisition of knowledge. However, some overdetermined problems, such as Problem 4, also have high didactic value. We believe that the procedure by which the six determined problems were derived from the overdetermined Problem 4 not only contributes to the conscious acquisition of mathematical knowledge, but it makes a significant contribution to developing the quality of students' thinking. This is especially true for flexibility and critical thinking: finding different ways to solve an overdetermined problem will inevitably reveal that it is overdetermined, and a critical analysis of the problem's conditions will result in identifying several problems, some of which are equivalent to the given one — as is the case with Problems 5–9, which are equivalent to Problem 4.

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